For the second class I posed two problems of Omar Khayyam. These are below : -

II. THE SECOND SPECIES. A CUBE AND A NUMBER ARE EQUAL TO SIDES.

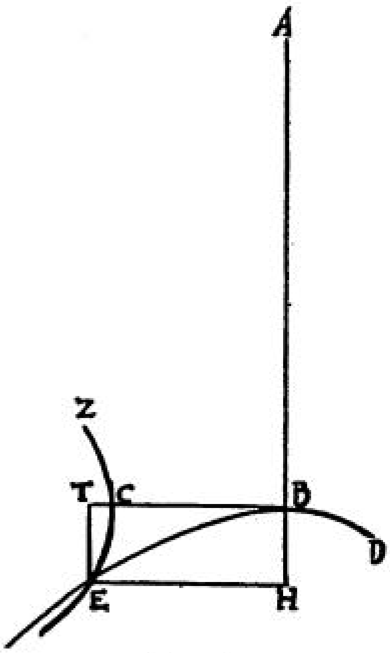

FIG. 18

Let the line AB (Fig.I8) be the side of a square equal to the number of the roots, and construct a solid having as its base the square of AB and equal to the given number, and let its height BC be perpendicular to AB. Describe a parabola having as its vertex the point B and its axis along the direction AB and its parameter AB. This is, then, the curve DBE, whose position is known. Construct also a second conic, namely, a hyperbola whose vertex is the point C and whose axis is arong the direction of BC. Each one of its two parameters, the perpendicular and the oblique, is equal to BC. It is the curve ECZ. This hyperbola also is known in position, as was shown by Apollonius in the 58th proposition of his first book. The two conics will either meet or will not meet. If they do not meet, the problem is impossible of solution. If they do meet, they do it tangentially at a point or by intersection at two points.

Suppose they meet at a point and let it be at E, whose position is known. Then drop from it two perpendiculars ET and EH on the two lines BT and BH. The two perpendiculars are known unerringly in position and magnitude. The line ET is an ordinate of the hyperbola. Consequently, the square of ET is to the product of BT and TC as the parameter is to the oblique, as was demonstrated by Apollonius in the twentieth proposition of the first book. The two sides, the perpendicular. and the oblique, are equal. Then the square ET is equal to the product of BT and TC, and BT to TE is as TE to TC. But the square of EH, which is equivalent to BT, is equal to the product of BH and BA, as was demonstrated in the second proposition of the first book of the treatise on conics.Consequently, AB is to BT as BT is to BH and as BH, which is equal to ET, is to TC. The four lines, AB, BT, ET, TC, then, are in continuous proportion, and the square of AB, the first, is to the square of BT, the second, as BT, the second, is to TC, the fourth. The cube of BT, then, is equal to the solid whose base is the square AB and whose height is CT. Let the solid whose base is the square of AB and whose height is BC, which was made equal to the given number, be added to both. Then the cube BT plus the given number is equal to the solid whose base is the square of AB and whose height is BT, which represents the number of the sides of the cube.

Thus it is shown that this species includes different cases and among its problems are some that are impossible. The species has been solved by means of the properties of the two conics, the parabola and the hyperbola.

III.THE THIRD SPECIES. A CUBE IS EQUAL TO SIDES PLUS A NUMBER.

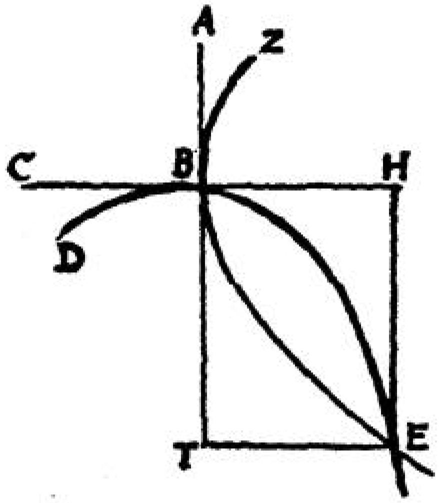

FIG. 19

Let AB (Fig.19), equivalent to the side of a square, which is equal to the number of the sides. Construct a solid whose base is the square of AB, and equal to the given number, and let its height BC be perpendicular on AB. Prolong AB and BC and describe a parabola whose vertex is the point B, whose axis is the prolongation of AB and whose parameter is AB. This is the parabola DBE, which is of known position and is tangent to the line BH as Apollonius has shown in the thirty-third proposition of the first book. Then describe another conic, a hyperbola whose vertex is the point B and whose axis is the prolongation of BC, having both of its parameters, the perpendicular and the oblique, equal to BC. This is the hyperbola ZBE, which is of known position, and tangent to the line AB. The two conics must necessarily intersect. Let them intersect at the point E whose position is known. Let fall from the point E the perpendiculars ET and EH. They are known in position and magnitude and the line EH is an ordinate of the hyperbola. As it was shown above, its square is equal to the product of CH and BH. Then CH to EH is as EH to HB, but EH, which is equal to BT, is to HB as ET is to AB, the parameter of the parabola. The four lines, then, are in continuous proportion. AB to HB is as HB to BT and as BT to CH. Then the square of AB, the first, to the square of HB, the second, is as HB, the second, to CH, the forth. The cube of HB, then, is equal to the solid whose base is the square of AB and whose height is CH, as both of their heights are reciprocally proportional to their bases. But this solid is equal to the solid whose base is the square of AB and whose height is BC. The solid was known to be equal to the given number plus the solid bounded by a base equal to the square of AB and a height BH, the latter being equal to the number of sides given to the cube BH. Then cube BH is equal to the given number plus the given number of sides, which was required.

It has been shown that this species has no varieties of cases and has no impossible problems. It was solved by means of the properties of the parabola combined with those of a hyperbola.